

# Cuspidal representations of $GL(n, F)$ distinguished by a maximal Levi subgroup, with $F$ a non-archimedean local field

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## Abstract

Let  $\rho$  is a cuspidal representation of  $GL(n, F)$ , with  $F$  a non archimedean local field, and  $H$  a maximal Levi subgroup of  $GL(n, F)$ . We show that if  $\rho$  is  $H$ -distinguished, then  $n$  is even, and  $H \simeq GL(n/2, F) \times GL(n/2, F)$ .

## 1 Preliminaries

Let  $F$  be nonarchimedean local field. We denote  $GL(n, F)$  by  $G_n$  for  $n \geq 1$ , and by  $N_n$  the unipotent radical of the Borel subgroup of  $G_n$  given by upper triangular matrices. For  $n \geq 2$  we denote by  $U_n$  the group of matrices  $u(x) = \begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix}$  for  $x$  in  $F^{n-1}$ .

For  $n > 1$ , the map  $g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$  is an embedding of the group  $G_{n-1}$  in  $G_n$ , we denote by  $P_n$  the subgroup  $G_{n-1}U_n$  of  $G_n$ .

We fix a nontrivial character  $\theta$  of  $(F, +)$ , and denote by  $\theta$  again the character  $n \mapsto \theta(\sum_{i=1}^{n-1} n_{i,i+1})$  of  $N_n$ . The normaliser of  $\theta|_{U_n}$  is then  $P_{n-1}$ .

When  $G$  is an  $l$ -group (locally compact totally disconnected group), we denote by  $Alg(G)$  the category of smooth complex  $G$ -modules. If  $(\pi, V)$  belongs to  $Alg(G)$ ,  $H$  is a closed subgroup of  $G$ , and  $\chi$  is a character of  $H$ , we denote by  $\delta_H$  the positive character of  $N_G(H)$  such that if  $\mu$  is a right Haar measure on  $H$ , and  $int$  is the action given by  $(int(n)f)(h) = f(n^{-1}hn)$ , of  $N_G(H)$  smooth functions  $f$  with compact support on  $H$ , then  $\mu \circ int(n) = \delta_H(n)\mu$  for  $n$  in  $N_G(H)$ .

If  $H$  is a closed subgroup of an  $l$ -group  $G$ , and  $(\rho, W)$  belongs to  $Alg(H)$ , we define the object  $(ind_H^G(\rho), V_c = ind_H^G(W))$  as follows. The space  $V_c$  is the space of smooth functions from  $G$  to  $W$ , fixed under right translation by the elements of a compact open subgroup  $U_f$  of  $G$ , satisfying  $f(hg) = \rho(h)f(g)$  for all  $h$  in  $H$  and  $g$  in  $G$ , and with support compact mod  $H$ . The action of  $G$  is by right translation on the functions.

If  $f$  is a function from  $G$  to another set, and  $g$  belongs to  $G$ , we will denote  $L(g)f : x \mapsto f(g^{-1}x)$  and  $R(g)f : x \mapsto f(xg)$ .

We say that a representation  $\pi$  of  $G$  is  $H$ -distinguished, if the complex vector space  $Hom_H(\pi, 1)$  is nonzero.

We will use the following functors following [B-Z]:

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- The functor  $\Phi^+$  from  $\text{Alg}(P_{k-1})$  to  $\text{Alg}(P_k)$  such that, for  $\pi$  in  $\text{Alg}(P_{k-1})$ , one has  $\Phi^+\pi = \text{ind}_{P_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes \theta)$ .
- The functor  $\Psi^+$  from  $\text{Alg}(G_{k-1})$  to  $\text{Alg}(P_k)$ , such that for  $\pi$  in  $\text{Alg}(G_{k-1})$ , one has  $\Psi^+\pi = \text{ind}_{G_{k-1}U_k}^{P_k}(\delta_{U_k}^{1/2}\pi \otimes 1) = \delta_{U_k}^{1/2}\pi \otimes 1$ .

We recall the following proposition, which is a consequence of theorem 4.4 of [B-Z].

**Proposition 1.1.** *Let  $\pi$  be a cuspidal representation of  $G_n$ , then the restriction  $\pi|_{P_n}$  is isomorphic to  $(\Phi^+)^{n-1}\Psi^+(1)$ .*

## 2 The result

Suppose  $n = p + q$ , with  $p \geq q \geq 1$ , we denote by  $M_{(p,q)}$  the standard Levi of  $G_n$  given by matrices  $\begin{pmatrix} h_p & \\ & h_q \end{pmatrix}$  with  $h_p \in G_p$  and  $h_q \in G_q$ , and by  $M_{(p,q-1)}$  the standard Levi of  $G_{n-1}$  given by matrices  $\begin{pmatrix} h_p & \\ & h_{q-1} \end{pmatrix}$  with  $h_p \in G_p$  and  $h_{q-1} \in G_{q-1}$ . We denote by  $M_{(p-1,q-1)}$  the standard Levi of  $G_{n-2}$  given by matrices  $\begin{pmatrix} h_{p-1} & \\ & h_{q-1} \end{pmatrix}$  with  $h_{p-1} \in G_{p-1}$  and  $h_{q-1} \in G_{q-1}$ .

Let  $w_{p,q}$  be the permutation matrix of  $G_n$  corresponding to the permutation

$$\begin{pmatrix} 1 & \dots & p-q & p-q+1 & p-q+2 & \dots & p-1 & p & p+1 & \dots & p+q-2 & p+q-1 & p+q \\ 1 & \dots & p-q & p-q+1 & p-q+3 & \dots & p+q-3 & p+q-1 & p-q+2 & \dots & p+q-4 & p+q-2 & p+q \end{pmatrix}$$

Let  $w_{p,q-1}$  be the permutation matrix of  $G_{n-1}$  corresponding to the permutation  $w_{p,q}$  restricted to  $\{1, \dots, n-1\}$ :

$$\begin{pmatrix} 1 & \dots & p-q & p-q+1 & p-q+2 & \dots & p-1 & p & p+1 & \dots & p+q-2 & p+q-1 \\ 1 & \dots & p-q & p-q+1 & p-q+3 & \dots & p+q-3 & p+q-1 & p-q+2 & \dots & p+q-4 & p+q-2 \end{pmatrix}$$

Let  $w_{p-1,q-1}$  be the permutation matrix of  $G_{n-2}$  corresponding to the permutation

$$\begin{pmatrix} 1 & \dots & p-q & p-q+1 & p-q+2 & \dots & p-2 & p-1 & p & \dots & p+q-3 & p+q-2 \\ 1 & \dots & p-q & p-q+1 & p-q+3 & \dots & p+q-5 & p+q-3 & p-q+2 & \dots & p+q-4 & p+q-2 \end{pmatrix}$$

We denote by  $H_{p,q}$  the subgroup  $w_{p,q}M_{(p,q)}w_{p,q}^{-1}$  of  $G_n$ , by  $H_{p,q-1}$  the subgroup  $w_{p,q-1}M_{(p,q-1)}w_{p,q-1}^{-1}$  of  $G_{n-1}$ , and by  $H_{p-1,q-1}$  the subgroup  $w_{p-1,q-1}M_{(p-1,q-1)}w_{p-1,q-1}^{-1}$  of  $G_{n-2}$ .

The two following lemmas and propositions are a straightforward adaptation of Lemma 1 and Proposition 1 of [K].

**Lemma 2.1.** *Let  $S_{p,q} = \{g \in G_{n-1}, \forall u \in U_n \cap H_{p,q}, \theta(gug^{-1}) = 1\}$ . Then  $S_{p,q} = P_{n-1}H_{p,q-1}$ .*

*Proof.* Denoting by  $L_{n-1}(g)$  the bottom row of  $g$ , one has  $\theta(gu(x)g^{-1}) = \theta(L_{n-1}(g).x)$  for  $u(x)$  in  $U_n$ . Hence  $\theta(gug^{-1}) = 1$  for all  $u$  in  $U_n \cap H_{p,q}$  if and only if  $g_{n-1,j} = 0$  for  $j = p-q, p-q+2, \dots, p+q-2$ . It is equivalent to say that  $g$  belongs to  $P_{n-1}H_{p,q-1}$ .  $\square$

**Lemma 2.2.** *Let  $S_{p,q-1} = \{g \in G_{n-2}, \forall u \in U_{n-1} \cap H_{p,q-1}, \theta(g^{-1}ug) = 1\}$ . Then  $S_{p,q} = P_{n-2}H_{p-1,q-1}$ .*

*Proof.* Denoting by  $L_{n-2}(g)$  the bottom row of  $g$ , and by  $u(x)$  the matrix  $\begin{pmatrix} I_{n-2} & x \\ 0 & 1 \end{pmatrix}$ , so that  $\theta(gug^{-1}) = \theta(L_{n-2}(g).x)$ . Hence  $\theta(gug^{-1}) = 1$  for all  $u$  in  $U_{n-1} \cap H_{p,q-1}$  if and only if  $g_{n-2,j} = 0$  for  $j = 0, 1, \dots, p-q, p-q+1$  and  $j = p-q+3, p-q+5, \dots, p+q-5, p+q-3$ . It is equivalent to say that  $g$  belongs to  $P_{n-2}H_{p-1,q-1}$ .  $\square$

**Proposition 2.1.** *Let  $\sigma$  belong to  $\text{Alg}(P_{n-1})$ , and  $\chi$  be a positive character of  $P_n \cap H_{p,q}$ , then there is a positive character  $\chi'$  of  $P_{n-1} \cap H_{p,q-1}$ , such that*

$$\text{Hom}_{P_n \cap H_{p,q}}(\Phi^+ \sigma, \chi) \hookrightarrow \text{Hom}_{P_{n-1} \cap H_{p,q-1}}(\sigma, \chi').$$

*Proof.* Let  $V$  be the space on which  $\sigma$  acts, and  $W = \phi^+ V$ . Let  $A$  the projection from  $\mathcal{C}_c^\infty(P_n, V)$  onto  $W$ , defined by  $A(f(p)) = \int_{P_{n-1}U_n} \delta_{U_n}^{-1/2}(y) \sigma(y^{-1}) f(yg) dy$ . Lifting through  $A$  gives a vector space injection of  $\text{Hom}_{P_n \cap H_{p,q}}(\Phi^+ \sigma, \chi)$  into the space of  $V$ -distributions  $T$  on  $P_n$  satisfying relations

$$T \circ R(h_0) = \chi(h_0) T \quad (1)$$

$$T \circ L(y_0) = \delta_{U_n}^{3/2}(y_0) T \circ \sigma(y_0) \quad (2)$$

for  $h_0$  in  $P_n \cap H_{p,q}$  and  $y_0 \in P_{n-1}U_n$ .

We introduce  $\Theta$  the map on  $P_n$  defined by  $\Theta(ug) = \theta(u)$  for  $u$  in  $U_n$  and  $g$  in  $G_{n-1}$ . Then the  $V$ -distribution  $\Theta.T$  is  $U_n$ -invariant, hence there is a  $V$ -distribution  $S$  with support in  $G_{n-1}$  such that  $\Theta.T = du \otimes S$  (where  $du$  denotes a Haar measure on  $U_n$ ), and thus  $T = \Theta^{-1}.du \otimes S$  has support  $U_n.\text{supp}(S)$ . It is easily verified that  $du \otimes S$  is invariant- $U_n$ , but because of relation (1),  $T$  is invariant- $(U_n \cap H_{p,q})$ . We deduce from these two facts that for  $g$  in  $\text{supp}(S)$ ,  $\Theta(gu)$  must be equal to  $\Theta(g)$  for any  $u$  in  $U_n \cap H_{p,q}$ . This means that  $\text{supp}(S) \subset S_{p,q}$ , and  $S_{p,q} = P_{n-1}H_{p,q-1}$  according to Lemma 2.1, hence  $T$  has support in  $P_{n-1}U_nH_{p,q-1}$ .

Now consider the projection  $B : \mathcal{C}_c^\infty(P_{n-1}U_n \times H_{p,q-1}, V) \rightarrow \mathcal{C}_c^\infty(P_{n-1}U_nH_{p,q-1}, V)$ , defined by  $B(\phi)(y^{-1}h) = \int_{P_{n-1} \cap H_{p,q-1}} \phi(ay, ah) da$  (which is well defined because of the equality  $P_{n-1}U_n \cap H_{p,q-1} = P_{n-1} \cap H_{p,q-1}$ ), and  $\phi \mapsto \tilde{\phi}$  the isomorphism of  $\mathcal{C}_c^\infty(P_{n-1}U_n \times H_{p,q-1}, V)$  defined by  $\tilde{\phi}(y, h) = \chi(h) \delta_{U_n}(y)^{3/2} \sigma(y) \phi(y, h)$ .

If one sets  $D(\phi) = T(B(\tilde{\phi}))$ , then  $D$  is a  $V$ -distribution on  $P_{n-1}U_n \times H_{p,q-1}$  which is invariant- $P_{n-1}U_n \times H_{p,q-1}$ . This implies that there exists a unique linear form  $\lambda$  on  $V$ , such that for all  $D(\phi) = \int_{P_{n-1}U_n \times H_{p,q-1}} \lambda(\phi(y, h)) dy dh$ .

Now for  $b$  in  $P_{n-1} \cap H_{p,q-1}$ , on has from the integral expression of  $D$ , the relation  $D \circ L(b, b) = \delta(b) D$  for some positive modulus character  $\delta$ . On the other hand, writing  $D$  as  $\phi \mapsto T(B(\tilde{\phi}))$ , one has  $\widetilde{L(b, b)\phi} = \chi(b) \delta_{U_n}^{3/2}(b) L(b, b) (\widetilde{\sigma(b^{-1})\phi})$  and  $B \circ L(b, b) = \delta_1(b) B$  for a positive modulus character  $\delta_1$ , sothat  $D \circ L(b, b) = \delta_1(b) \chi(b) \delta_{U_n}^{3/2}(b) D \circ \sigma(b^{-1})$ . Comparing the two expressions for  $D \circ L(b, b)$ , we get the relation  $D \circ \sigma(b) = \chi'(b) D$ , with  $\chi'$  being the positive character  $\delta^{-1} \delta_1 \chi \delta_{U_n}^{3/2}$  of  $P_{n-1} \cap H_{p,q-1}$ .

This in turn implies that the linear form  $\lambda$  on  $V$  satisfies the same relation, i.e. belongs to  $\text{Hom}_{P_{n-1} \cap H_{p,q-1}}(\sigma, \chi')$ , and  $T \mapsto \lambda$  gives a linear injection of  $\text{Hom}_{P_n \cap H_{p,q}}(\Phi^+ \sigma, \chi)$  into  $\text{Hom}_{P_{n-1} \cap H_{p,q-1}}(\sigma, \chi')$ , and this proves the proposition.  $\square$

Using Lemma 2.2 instead of Lemma 2.1 in the previous proof, one obtains the following statement.

**Proposition 2.2.** *Let  $\sigma'$  belong to  $\text{Alg}(P_{n-2})$ , and  $\chi'$  be a positive character of  $P_{n-1} \cap H_{p,q-1}$ , then there is a positive character  $\chi''$  of  $P_{n-2} \cap H_{p-1,q-1}$ , such that*

$$\text{Hom}_{P_{n-1} \cap H_{p,q-1}}(\Phi^+ \sigma', \chi') \hookrightarrow \text{Hom}_{P_{n-2} \cap H_{p-1,q-1}}(\sigma, \chi'').$$

A consequence of these two propositions is the following.

**Proposition 2.3.** *Let  $n \geq 3$ , and  $p$  and  $q$  two integers with  $p + q = n$  and  $p - 1 \geq q \geq 0$ , then one has  $\text{Hom}_{P_n \cap H_{p,q}}((\Phi^+)^{n-1} \Psi^+(1), 1) = 0$ .*

*Proof.* Using repeatedly the last two propositions, we get the existence of a positive character  $\chi$  of  $P_{p-q+1}$  such that  $\text{Hom}_{P_n \cap H_{p,q}}((\Phi^+)^{n-1}\Psi^+(1), 1) \hookrightarrow \text{Hom}_{P_{p-q+1} \cap H_{p-q+1,0}}((\Phi^+)^{p-q}\Psi^+(1), \chi) = \text{Hom}_{P_{p-q+1}}((\Phi^+)^{p-q}\Psi^+(1), \chi)$ , and this last space is 0 because  $(\Phi^+)^{p-q}\Psi^+(1)$  and  $\chi$  are two non isomorphic irreducible representations of  $P_{p-q+1}$ , according to corollary 3.5 of [B-Z].  $\square$

This implies the following theorem about cuspidal representations.

**Theorem 2.1.** *Let  $\pi$  be a cuspidal representation of  $G_n$ , which is distinguished by a maximal Levi subgroup  $M$ , then  $n$  is even and  $M \simeq M_{n/2, n/2}$ .*

*Proof.* Let  $M$  be the maximal Levi subgroup such that  $\pi$  is  $M$ -distinguished. Then  $M$  is conjugate to a standard Levi subgroup  $M_{p,q}$  with  $p \geq q$  and  $p+q = n$ . Suppose  $p \geq q+1$ ,  $M_{p,q}$  is conjugate to  $H_{p,q}$ , so that  $\pi$  is  $H_{p,q}$ -distinguished, and  $\pi|_{P_n}$  is thus  $H_{p,q} \cap P_n$ -distinguished. But by Proposition 1.1, the restriction  $\pi|_{P_n}$  is isomorphic to  $(\Phi^+)^{n-1}\Psi^+(1)$ , and this contradicts Proposition 2.3. Hence one must have  $p = q$ , and this proves the theorem.  $\square$

## References

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